INFORMATION AND CONTROL 24, 384–388 (1974)

A Note on Perimeter and Diameter in Digital Pictures*

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It is shown that the "intrinsic diameter" of a connected digital object C—i.e., the maximum distance in C between any two points of C—never exceeds half of C's total perimeter (counting all of C's borders, including hole borders). As a preliminary result, it is shown that a simply connected digital object is a simple arc if and only if it has at most two "end" or "corner" points (and, in fact, it can then only have two "end" points).

Let C be a connected component of 1's in a binary-valued digital picture. (See [1–3] for detailed definitions of the concepts used in this note, as well as for the earlier results that are used here.) For each connected component D of 0's that is adjacent to C, let (C, D) be the D-border of C, i.e., the set of points of C that are adjacent to points of D. From now on, we assume that C is 4-connected and D 8-connected; the proofs in the opposite case are analogous.

By the *perimeter* $p(C, D)$ of (C, D) we mean the number of moves from one point of C to the next that are made by the standard border following algorithm when it follows (C, D). (The algorithm is to be interpreted here so that successive points of C that it visits are always 4-neighbors, some of which may only be 8-adjacent to D.) By the *total perimeter* $p(C)$ of C we mean the sum of the perimeters of all the (C, D)'s.

For any pair of points x, y of C, let $d(x, y)$ be the 4-neighbor *distance* from x to y in C, i.e., the length of the shortest 4-path from x to y that lies in C. (If we were using 8-connectedness for C, we would use 8-neighbor distance here instead.) By the *intrinsic diameter* $d(C)$ of C we mean the greatest such distance, for any pair of points of C. A pair of points in C whose distance is equal to $d(C)$ will be called "diametrically opposite."

* This is Technical Report TR-252.

† The support of the Office of Computing Activities, National Science Foundation, under Grant GJ-32258X, is gratefully acknowledged, as is the help of Mrs. Eleanor B. Waters in preparing this paper. The author also thanks Prof. E. R. Caianiello for providing the pleasant environment in which this work was done, and Dr. S. Levialdi for many helpful discussions.

Our main goal in this note is to prove

THEOREM 1. $d(C) \leqslant (1/2)\, p(C)$.

Proof. We shall show at the end of this note that the theorem is true when C is simply connected; we proceed by induction on the number of holes in C. Let x, y be diametrically opposite points of C, and let C' denote C with one of its holes, say H, filled in. Thus $p(C) = p(C') + p(C, H)$.

Let d and d' denote, respectively, the distances between x and y in C and in C'. Let ρ' be a path of length d' from x to y in C', and let u and v be the first and last points in which ρ' meets (C, H). Evidently, there exists a path σ from u to v in C whose length is at most $(1/2)\, p(C, H)$—indeed, we can construct σ by simply following the border (C, H) the shorter way around from u to v. If we replace the part of ρ' between u and v by σ, we obtain a path ρ from x to y in C. Moreover, the length r of ρ is at most $(1/2)\, p(C, H)$ greater than the length d' of ρ'. Hence,

$$\begin{aligned} d(C) = d \leqslant r \leqslant d' + (1/2)\, p(C, H) &\leqslant (1/2)\, p(C') + (1/2)\, p(C, H) \\ &= (1/2)\, p(C). \quad \blacksquare \end{aligned}$$

It should be noted that d can be much smaller than p. For example, let C consist of a horizontal line $2n + 1$ long that makes T-junctions with n vertical lines, each m long. Then $p(C)$ is greater than $2mn$, but $d(C)$ is only about $2m + n$.

To prove Theorem 1 in the simply connected case, we shall first establish a result about simply connected objects that have very few "deletable" points. It is known [1, 2] that any such object, having more than one point, has at least two "end" points, or an "end" and two "corners," or four "corners"; both of these types of points are "deletable" in the sense that their removal leaves the object simply connected. We shall now prove

THEOREM 2. *A simply connected object S, having at least two points, has exactly two deletable points (i.e., ends and corners) if and only if S is a simple arc.*

This result is somewhat analogous to Theorem 4.4 of [2], which states that an object with exactly one hole has no deletable points if and only if it is a simple closed curve.

Proof. "If" is clear, since an arc (having more than one point) has exactly two deletable points, namely, its endpoints. To prove "only if", we note first that by the remarks just above, if S has exactly two deletable points, they must both be "ends." The assertion is trivial if S has just two points; we proceed by induction on the number of points in S.

Let x be the leftmost of the uppermost points of S, so that the neighborhood of x looks like

$$\begin{matrix} y & 0 & z \\ 0 & x & a \\ b & c & d, \end{matrix}$$

where $x = 1$; y can be either 0 or 1; and $z = 0$ if $a = 1$. Since S consists of more than just the single point x, at least one of a and c must be 1. Since S has no corners, if a and c are both 1, d must be 0.

(a) Suppose first that $a = 1$, $c = d = 0$, i.e., that the neighborhood is

$$\begin{matrix} y & 0 & 0 \\ 0 & 1 & 1 \\ b & 0 & 0 \end{matrix}$$

(if $c = 1$, $a = b = d = 0$, i.e., we have

$$\begin{matrix} y & 0 & z \\ 0 & 1 & 0 \\ 0 & 1 & 0, \end{matrix}$$

then the proof is exactly analogous). If we delete x from S, we obtain an S' that is still simply connected. The only point of S' that could be an end in S' but not in S is a itself; and no point could be a corner in S' but not in S. Hence, S' can only have two ends, one of which is a, and by induction hypothesis, this makes S' an arc. (The only other possibility is that S' consists of the single point a.) Thus, S too is an arc, obtained by adding the point x to one end of S'.

(b) Suppose that $a = d = 1$, $c = 0$ (or $b = c = 1$, $a = d = 0$; or $c = d = 1$, $a = b = 0$)—i.e., that we have

$$\begin{matrix} y & 0 & 0 \\ 0 & 1 & 1 \\ b & 0 & 1 \end{matrix} \quad \left(\text{or} \quad \begin{matrix} y & 0 & z \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \quad \text{or} \quad \begin{matrix} y & 0 & z \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}\right).$$

In this case (and analogously in the other two cases), if d is an end or corner in S', it is also one in S; thus the only point that can be an end or corner in S' but not in S is a itself. Hence, here again, S' can only have two ends, one of which is a. This makes S' an arc, and S is then an arc also.

(c) If $a = c = 1$, $b = d = 0$, we have

$$\begin{matrix} y & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0, \end{matrix}$$

so that x is not a deletable point, and if we delete it, S breaks up into two simply connected components, call them S_a and S_c, each having fewer points than S. If either of these has more than one point, then by the same argument as before, it has exactly two ends and, hence, is an arc. Thus, in any case, S is obtained by joining two arcs (one or both of which may be single points) through the added point x, which makes S an arc here too.

(d) Similarly, if $a = b = c = 1$, $d = 0$, we have

$$\begin{matrix} y & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0. \end{matrix}$$

Here, only $\underline{a}$ can be an end in S_a but not in S, and only c can be an end or corner in S_c but not in S—hence, it cannot be a corner, since S would then have three deletable points (two from S_c, one from S_a). Thus, here again S_a and S_c each have just two ends (or S_a is a single point), making them arcs, and thus making S an arc, as before.

(e) Finally, if $b = c = d = 1$, $a = 0$, we have

$$\begin{matrix} y & 0 & z \\ 0 & 1 & 0 \\ 1 & 1 & 1. \end{matrix}$$

Here no point can be an end or corner in S' without also being one in S. Since S' has at least two such points, this gives S three (those of S', together with x), a contradiction. Thus this case cannot arise, and the proof is complete. ▮

We can now prove Theorem 1 in the simply connected case. The assertion is trivial if C has only one or two points; we proceed by induction on the number of its points. Let x, y be diametrically opposite points of C. If x and y are the only ends or corners of C, then by Theorem 2, C is an arc; thus, $p(C) = 2n - 1$, and $d(C) = n - 1 < (1/2)\,p(C)$, where n is the number of points of C.

Otherwise, C has an end or corner z other than x and y. Readily, deleting z cannot change the distance between any other two points (since a path through an end cannot be a shortest path, and a path through a corner can be diverted to avoid the corner without changing its length). Hence, if C' is the result of deleting z from C, we have $d(C') = d(C) = d(x, y)$. Moreover, the perimeter of C' is no greater than that of C, since deleting an end shortens the perimeter by 2, while deleting a corner does not change the perimeter. Hence,

$$d(C) = d(C') \leqslant (1/2)\, p(C') \leqslant (1/2)\, p(C),$$

which completes the proof of Theorem 1.

RECEIVED: August 27, 1972; REVISED: October 10, 1973.

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